# Assignment 2.

This assignment is due February 28th. If you need more time, ask for an extension (just don't get overwhelmed by homeworks piling up).

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper.

(1) (a) Let $V$ be the vector space of all $2 \times 2$ matrices over the field $F$. Prove that $V$ has dimension 4 by exhibiting a basis for $V$ which has four elements.

(b) Let $V$ be as in the previous item. Let $W_1$ be the set of matrices of the form

$$\begin{pmatrix} x & -x \\ y & z \end{pmatrix}$$

and let $W_2$ be the set of matrices of the form

$$\begin{pmatrix} a & b \\ -a & c \end{pmatrix}.$$

Prove that $W_1$ and $W_2$ are subspaces of $V$.

(c) Find the dimensions of $W_1$, $W_2$, $W_1 + W_2$ and $W_1 \cap W_2$.

(2) Let $V$ be a vector space over a subfield $F$ of the complex numbers. Suppose $\alpha, \beta, \gamma$ are linearly independent vectors in $V$. Prove that $(\alpha+\beta), (\beta+\gamma), (\gamma+\alpha)$ are linearly independent.

(3) Let vector space $V$ be spanned by vectors $\alpha_1, \alpha_2, \ldots, \alpha_k$. Prove that there is a basis for $V$ which is a *subset* of $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$.

(4) Let $W$ be subspace of $\mathbb{C}^3$ spanned by $\alpha_1 = (1, 0, i)$ and $\alpha_2 = (1 + i, 1, -1)$.

(a) Show that $\alpha_1$ and $\alpha_2$ form a basis for $W$.

(b) Show that the vectors $\beta_1 = (1, 1, 0)$ and $\beta_2 = (1, i, 1 + i)$ are in $W$ and form another basis for $W$.

(c) What are coordinates of $\alpha_1$ and $\alpha_2$ in the ordered basis $\{\beta_1, \beta_2\}$ for $W$?

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(5) *(Solving a linear recurrence)* In this problem the goal is to find an explicit formula for $n$th Fibonacci number $f_n$.

Let $V$ be the set of sequences $(x_n)$ $(n = 1, 2, \ldots)$ in $\mathbb{R}$ that satisfy

$$x_{n+2} = x_{n+1} + x_n$$

for each integer $n \geq 1$.

(a) Prove that $V$ is a vector space over $\mathbb{R}$ with respect to term-wise addition and multiplication.
(b) Prove that $\dim V = 2$ (Hint: prove that two sequences $X$ with $x_1 = 1, x_2 = 0$ and $Y$ with $y_1 = 0, y_2 = 1$ form a basis for $V$).
(c) Find two geometric series that belong to $V$. (That is, find $\lambda_1, \lambda_2$ such that sequence $(1, \lambda, \lambda^2, \lambda^3, \ldots) \in V$ for $\lambda = \lambda_1$ and $\lambda = \lambda_2$).
(d) Let $X_1$, $X_2$ be the two geometric series from the previous item. Prove that $(X_1, X_2)$ is a basis for $V$.
(e) Find coordinates of the Fibonacci sequence $(f_n)$ in this basis (Fibonacci sequence $(f_n)$ is an element of $V$ with $f_1 = f_2 = 1$). Find an explicit formula for $n$th Fibonacci number $f_n$.

(6) *(Solving a linear recurrence, continued)* In this problem the goal is to find an explicit formula for $n$th term of an arbitrary linear recurrent sequence of degree 2 (that is, of an arbitrary sequence where $x_{n+2}$ is a fixed linear combination of the two previous terms).

Let $V_{a,b}$ be the linear space of sequences $(x_n)$ $(n = 1, 2, \ldots)$ in $\mathbb{R}$ that satisfy

$$x_{n+2} = ax_{n+1} + bx_n$$

for each integer $n \geq 1$, where $a, b \in \mathbb{R}$. Consider quadratic polynomial $\chi(t) = t^2 - at - b$.

(a) Suppose $\chi$ has two distinct real roots $\lambda_1, \lambda_2$. Show that sequences $(\lambda_1^{n-1})$ and $(\lambda_2^{n-1})$ form a basis for $V_{a,b}$.
(b) Suppose $\chi$ has two distinct complex (with non-zero imaginary part) roots

$$\lambda_1 = \lambda(\cos\mu + i\sin\mu), \quad \lambda_2 = \lambda(\cos\mu - i\sin\mu)$$

Prove that $(\lambda^{n-1}\cos(n-1)\mu)$, $(\lambda^{n-1}\sin(n-1)\mu)$ belong to $V_{a,b}$ and form a basis for $V_{a,b}$.
*Hint*: to make computation shorter, show that $(\lambda_1^{n-1} + \lambda_2^{n-1})$ and $-i(\lambda_1^{n-1} - \lambda_2^{n-1})$ belong to $V_{a,b}$. Express the latter two sequences in terms of $\lambda, \mu$ using de Moivre's formula

$$(A(\cos\varphi + i\sin\varphi))^n = A^n(\cos n\varphi + i\sin n\varphi).$$

(c) Suppose $\chi$ has a double root $\lambda_1$. Show that sequences $(\lambda_1^{n-1})$, $(n\lambda_1^{n-1})$ form a basis for $V_{a,b}$.
*A comment about item (b): note that sequences $(\lambda_1^{n-1})$ and $(\lambda_2^{n-1})$, while satisfying the recurrent relation, do not belong to $V_{a,b}$ because they consist of complex numbers. So, while one could, surely, use them to derive explicit formula for $n$th term of arbitrary $v \in V_{a,b}$, that would give an answer in terms different than those of the question. We go through all the trouble with sines and cosines to give a real-numbers answer to a real-numbers question.*

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(7) Consider $F[x]$, the linear space of polynomials over field $F$. Let $D : F[x] \to F[x]$ be differentiation operator defined by

$$D(a_n x^n + \dots + a_1 x + a_0) = n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} \dots + a_1.$$

(a) Prove the product rule. That is, prove that for any $f, g \in F[x]$,

$$D(fg) = D(f)g + fD(g).$$

(Hint: the equality to prove is linear in $f$ and linear in $g$.)

(b) Let $\partial : F[x] \to F[x]$ be *some* differentiation operator, i.e. a linear transformation that satisfies product rule

$$\partial(fg) = \partial(f)g + f\partial(g).$$

Find all possible operators $\partial$. (Hint: start by assuming $\partial(x) = f_0$. Then find $\partial(x^2), \partial(x^3), \dots$ using product rule.)

(c) Let $X$ be the operator of multiplication by $x$:

$$X(a_n x^n + \dots + a_1 x + a_0) = a_n x^{n+1} + \dots + a_1 x^2 + a_0 x.$$

Prove that

$$DX - XD = I,$$

i.e. that for any polynomial $f \in F[x]$,

$$DX(f) - XD(f) = f.$$

(8) Questions to ponder. (Do not submit anything for this problem in written. Just think about these when you have time. However, these questions do have quite specific answers.)

(a) How much changes compared to Problem 6 if we consider higher degree linear recurrences? (That is, if we consider space $V$ of all sequences in $\mathbb{R}$ that satisfy $x_{n+k+1} = a_k x_{n+k} + a_{k-1} x_{n+k-1} + \dots + a_1 x_{n+1}$.) For example, what basis vectors correspond to a root of multiplicity $> 2$?

(b) Recall how one solves an ordinary linear differential equation with constant coefficients. It should awfully remind you of what was going on in Problems 5 and 6. Where does this similarity come from, exactly?

(c) In Problem 2, $F$ is a subfield of complex numbers. What goes wrong in case of field on two elements? Is it possible to construct a counterexample to Problem 2 in case of such field?